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Uni versal Journal of Applied Mathematics 2(6): 250-255, 2014 DOI: 10.13189/ujam.2014.020603 http://www.hrpub.org The Scrambling Index of Two-colored Wielandt Digraph Mulyono , Saib Suwilo * Department of Mathematics, University of Sumatera Utara, Medan 20155, Indonesia * Corresponding Author: saib@usu.ac.id

Abstract A digraph is primitive provided there is a positive integer k such that for each pair of vertices u and v there exist walks of length k from u to v and from v to u . The scrambling index of a primitive digraph D is the smallest positive integer k such that for each pair of vertices u and v in D there is a vertex w such that there exist walks of length k from u to w and from v to w . A two-colored digraph is a digraph each of whose arc is colored by red or blue.

In this paper we generalize the notion of scrambling index of a primitive digraph to that of two-colored digraph. We de?ne the scrambling index of a two-colored digraph D (2) to be the smallest positive integer h + l over all pairs of nonnegative integers (h,ÿl) such that for each pair of distinct vertices u and v there is a vertex w with the property that there are walks form u to w and from v to w consisting of h red arcs and l blue arcs. For two-colored Wielandt digraph on n = 4 vertices we show the scrambling index lies on the interval [n 2 - 3n +ÿ3 ,ÿn 2 - 2n +ÿ2] .

Keywords Two-colored digraph, Primitive digraph, Scrambling Index, Wielandt digraph 1 Introduction By a nonnegative integer vector x = 0 we meant a vector each of whose entry is a nonnegative integer. Therefore, the notion z = x means that z - x = 0. Let D be a digraph. A walk of length k from u to v is a sequence of arcs of the form u = v 0 ? v 1 ,ÿv 1 ? v 2 ,ÿ.ÿ.ÿ.ÿ,ÿv k -1 ? v k = v .

We use the notation u k ? v walk to represent a walk of length k from u to v . A u ? v path is a walk with distinct vertices except possibly u = v . A cycle is a u ? v path with u = v . A digraph D is strongly con- nected if for each pair of vertices u and v there is a u ? v walk and a v ? u walk. A strongly connected digraph D is primitive provided there is a positive integer k such that for each pair of vertices u and v there exist a u k ? v walk and a v k ? u walk. The smallest of such positive integer k is the exponent of D and is denoted by exp(D).

It is a well known result that for a primitive digraph on n vertices, see [4], the exp(D) = (n - 1) 2 +ÿ1 . The upper bound is achieved the Wielandt digraph W n on n vertices that is a digraph consists of a Hamiltonian cycle v 1 ? v 2 ?ÿ·ÿ·ÿ·ÿ? v n ? v 1 and the arc v n - 1 ? v 1 as in Figure 1. The notion of scrambling index of a primitive digraph was ?rst introduced by Akelbek and Kirkland [1, 2].

They de- ?ne the scrambling index of a primitive digraph D to be the smallest positive integer k such that for every pair of vertices u and v in D there exists a vertex w in D such that there is a u k ? w walk and a v k ? w walk. The scrambling index of a primitive digraph is denoted by k (D). Their results, see [2], show that primitive digraph with largest scrambling index is achieved by the Wielandt digraph.

• v n - 1 _ H H H H H H Y _ _ _ _ • v n- 2 _ _ _ _ • v n- 3 _ _3 ·ÿ·ÿ· Q Qs • v 3 A A AU • v 2 C C CW • v 1 _ _ _ _ _ _ v n • Figure 1. The Wielandt Digraph W n A two-colored digraph is a digraph each of whose arc is colored by red or blue. For nonnegative integers h and l , an (h,ÿl)-walk in a two-colored digraph is a walk consisting of h red arcs and l blue arcs.

An (h,ÿl)-walk from u to v is denoted by u (h,l) -? v . For a walk W in D (2) , we denote r (W) and b (W) respectively to be the number of red arcs and blue arcs of W . The vector [r (W) b(W)] is the composition of W .

A strongly connected two-colored digraph D (2) is primitive provided that there are nonnegative integers h and l such that for each pair of vertices u and v in D (2) there exist a u (h,l) -? v walk and a v (h,l) -? u walk. Let D (2) be a two-colored digraph and let C = {C 1 ,ÿC 2 ,ÿ.ÿ.ÿ.ÿ,ÿC q } be the set of all cycles in D (2) .

De?ne the cycle matrix M of D (2) to be the matrix M = [r (C 1) r (C 2) ·ÿ·ÿ· r (C q) b (C 1) b(C 2) ·ÿ·ÿ· b (C q)] . That is M is the matrix such that the ith column of M is the composition of the cycle C i , i =ÿ1, 2,ÿ.ÿ.ÿ.ÿ,ÿq . The content of Universal Journal of Applied Mathematics 2(6): 250-255, 2014 251 the 2 by q matrix M is de?ned to be 0 if the rank of M is less than 2 and the greatest common divisor of the determinants of the 2 by 2 submatrices of M , otherwise.

The following theorem presents an algebraic characterization for a primitive two-colored digraph. Theorem 1.1 [5] Let D (2) be a two-colored digraph with at least one arc of each color. The two-colored digraph D (2) is primitive if and only if the content of its cycle matrix is 1 .

We generalize the notion of scrambling index of a primi- tive digraph to that of scrambling index of a primitive two- colored digraph. For a primitive two-colored digraph D (2) we de?ne the scrambling index of D (2) to be the smallest positive integer h + l over all nonnegative integers h and l such that for every pair of vertices u and v in D (2) there is a vertex w with the property that there is a u (h,l) -? w walk and a v (h,l) -? w walk.

The scrambling index of D (2) is denoted by k (D (2)) . Ananichev, Gusev, and Volkov [3] have used primitive di- graphs with large exponents in attempt to ?nd slowly syn- chronizing automata. Such primitive digraphs consist of cy- cles with two distinct lengths.

An automaton on two input let- ters over a ?nite states is synchronizing if there exists a word, called a reset word, of ?nite length that brings all states to a particular state. ? Cern ´ y's conjecture states that for an automa- ton on two input letters A with n states, the length of a reset word is no more than (n - 1) 2 .

This is close to the exponent of a Wielandt digraph on n vertices which is (n - 1) 2 +ÿ1 . Let A be a synchronizing automaton on two input letters and let D (2) be a two-colored digraph representation of A . An automaton A is synchronizing with reset word of length h + l if there exists a vertex u in D (2) such that for each vertex v in D (2) there is a v (h,l) -? u walk, moreover the order of appearance of red and blue arcs in each v ? u walk are the same.

Thus the scrambling index of a two-colored digraph may be used as a lower bound for t he length of a reset word for a synchronizing automaton with two input letters. In this paper, we discuss the scrambling index of two- colored Wielandt digraphs W (2) n that is a two-colored digraph obtained by coloring each arc of the Wielandt digraph W n with either red or blue.

In Section 2, we discuss a way to determine a lower and an upper bound for scrambling index of two-colored digraph consisting two cycles. In Section 3 we discuss the scrambling index of two-colored Wielandt di- graph. 2 Lower and Upper Bound In this section, we discuss a way in setting up lower and upper bound for scrambling index of primitive two-colored digraph, especially those that consist of two cycles. We ?rst note that every walk in a two-colored digraph can be decomposed into a path and some cycles.

This implies for every u (h,l) -? v walk we have the following relationship [h l] = [r (p uv) b (p uv)] + z 1 [r (C 1) b (C 1)] + z 2 [r (C 2) b (C 2)] + ·ÿ·ÿ· + z q [r (C q) b (C q)] = [r (p uv) b (p uv)] + M z for some path p uv from u to v and some nonnegative integer vector z . The following proposition will be useful in order to deter- mine an upper bound for scrambling index. Proposition 2.1

Let D (2) be a primitive two-colored di- graph consisting of two cycles C 1 and C 2 . Suppose v is a vertex that belongs to both cycles. If for some positive inte- gers h and l , there is a path p u,v from u to v such that the system M z + [r (p u,v) b (p u,v)] = [h l] (1) has nonnegative integer solution, then there is an (h,ÿl) -walk from u to v . Proof. Assume that the solution to the system (1) is z = (z 1 ,ÿz 2) T . We consider four cases.

If z 1 > 0 and z 2 > 0 , then the walk that starts at u , moves to v along the (r (p u,v) ,ÿb (p u,v)) -path p u,v and ?nally moves z 1 and z 2 times around the cycles C 1 and C 2 , respectively, and back at v is an (h,ÿl) -walk from u to v . If z 1 =ÿ0 and z 2 > 0 , then the walk that starts at u , moves to v along the (r (p u,v) ,ÿb (p u,v)) -path p u,v and ?nally moves z 2 times around the cycle C 2 and back at v is an (h,ÿl) -walk from u to v .

Similarly if z 1 > 0 and z 2 =ÿ0 , then the walk that starts at u , moves to v along the (r (p u,v) ,ÿb (p u,v)) -path p u,v and ?nally moves z 1 times around the cycle C 1 and back at v is an (h,ÿl) -walk from u to v . Finally, if z 1 = z 2 =ÿ0 , then the (r (p u,v) ,ÿb (p u,v)) -path p u,v from u to v is an (h,ÿl) -walk.

We next discuss a way in setting up a lower bound for the scrambling index. Let u and v be two different vertices in a primitive two-colored digraph D (2) . For a vertex w in D (2) , the local scrambling index of u and v at the vertex w , k u,v (w) , is the smallest positive integer h + l over all pairs of nonnegative integers h and l such that there are u (h,l) -? w and v (h,l) -? w walks.

The local scrambling index of vertices u and v in D (2) , denoted k u,v (D (2)) , is de?ned by k u,v (D (2))ÿ=ÿmin w { k u,v (w) } . From the de?nition of scrambling index we have max u,v ? V (D (2)) { k u,v (D (2)) }ÿ= k (D (2)) . (2) Let D (2) be a primitive two-colored digraph consisting of two cycles and let u and v be two distinct vertices in D (2) .

For some vertex w suppose that k u,v (w) is obtained by an (h,ÿl) -walk. We have the following result that will be useful in ?nding a lower bound for k u,v (D (2)) and hence for the scrambling index. Lemma 2.2 Let D (2) be a primitive two-colored digraph consisting of two cycles C 1 and C 2 with cycle matrix M = [r (C 1) r (C 2) b (C 1) b (C 2)] , and let u and v be any two dis- tinct vertices in D (2) . Suppose there is a vertex w such that there is a u (h,l) -? w walk and v (h,l) -? w walk.

If q 1 and q 2 are integers such that [h l] = M [q 1 q 2] , then [q 1 q 2] = M - 1 [r (p uw) b (p uw)] for some path p uw , and [q 1 q 2] = M - 1 [r (p vÿw) b (p vÿw)] for some path p vÿw . 252 The Scrambling Index of Two-colored Wielandt Digraph Proof. Since D (2) is primitive, then by Theorem 1.1 we have det(M)ÿ= ± 1 . Without loss of generality we assume that det(M)ÿ=ÿ1 .

Since every walk can be decomposed into a path and some cycles, then [h l] = [r (p uw) b (p uw)] + M z , (3) for some path p uw from u to w and some nonnegative integer vector z . Comparing (3) and [h l] = M [q 1 q 2] , we have z = [q 1 q 2] - M - 1 [r (p uw) b (p uw)] = 0 . Hence [q 1 q 2] = M - 1 [r (p uw) b (p uw)] for some path p uw . Similarly [q 1 q 2] = M - 1 [r (p vÿw) b (p vÿw)] for some path p vÿw .

We note from Lemma 2.2 that [q 1 q 2] = M - 1 [r (p uw) b (p uw)] = [b (C 2) r (p uw) - r (C 2) b (p uw) r (C 1) b (p uw) - b (C 1) r (p uw)] . Hence we have q 1 = b (C 2) r (p uw) - r (C 2) b (p uw) (4) for some path p uw from u to w .

Similarly, we have q 2 = r (C 1) b (p vÿw) - b (C 1) r (p vÿw) (5) for some path p vÿw from v to w . Thus [h l] = M [b (C 2) r (p uw) - r (C 2) b (p uw) r (C 1) b (p vÿw) - b (C 1) r (p vÿw)] for some paths p uw and p vÿw . 3 Main Results In this section we present formulae for scrambling index of two-colored Wieland digraph.

We ?rst present primitivity condition for two-colored Wielandt digraph and then discuss formulae their scrambling index. We note that the Wielandt digraph consists of two cycles. They are the n -cycle v 1 ? v 2 ?ÿ·ÿ·ÿ·ÿ? v n ? v 1 and the (n - 1) -cycle v 1 ? v 2 ?ÿ·ÿ·ÿ·ÿ? v n - 1 ? v 1 . As a consequence of Theorem 1.1

we have the following charac- terization for primitivity of a two-colored Wielandt digraph. Lemma 3.1 [6] A two-colored Wielandt digraph W (2) n on n vertices is primitive if and only if its cycle matrix M = [r (C 1) r (C 2) b (C 1) b (C 2)] = [n - 1 n - 2 1ÿ1] . Lemma 3.1 implies that a primitive two-colored Wielandt digraph has at most two blue arcs. Moreover, every cycle contains exactly one blue arc.

We determine the scrambling index of W (2) n based on how many blue arcs W (2) n has. If W (2) n has only one blue arcs, then the blue arc must lie on the v 1 ? v n - 2 path. So the blue arc of W (2) n must be of the form v a ? v a +1 where 1 = a = n - 2 .

If w (2) n has two blue arcs, then one of them must lie on the cycle C 2 but not on C 1 and the other must lie on C 1 but not on C 2 . This implies the two blue arcs either have the same terminal vertex or have the same initial vertex. We ?rst discuss the case where W (2) n has only one blue arc and then discuss the case where W (2) n has two blue arcs. Theorem 3.2

Let W (2) n be a two colored Wielandt digraph on n = 4 vertices. If W (2) n has only one blue arc v a ? v a +1 , where 1 = a = n - 2 , then k (W (2) n)ÿ= n 2 - 2 n +ÿ1 - a . Proof. We show that k (W (2) n) = n 2 - 2 n +ÿ1 - a . This is done by showing that k v a ,v a +1 (W (2) n) = n 2 - 2 n +ÿ1 - a .

We assume that there are v a (h,l) -? w and v a +1 (h,l) -? w walks for some vertex w ? W (2) n . We present a lower bound for k v a ,v a +1 (w) and consider two cases depending on the position of the vertex w . Case 1 . The vertex w = v t where 1 = t = a Notice that there are two paths p a +1 ,t from v a +1 to v t .

They are an (n - 2ÿ+ t - a, 0) -path and an (n - 1ÿ+ t - a) -path. Considering the (n - 2ÿ+ t - a, 0) -path and (4) we have q 1 = b (C 2) r (p a +1 ,t) - r (C 2) b (p a +1 ,t) =ÿ(1)(n - 2ÿ+ t - a) - (n - 2)(0)ÿ= n - 2ÿ+ t - a.

Considering the (n - 1ÿ+ t - a, 0) -path and (4) we have q 1 = b (C 2) r (p a +1 ,t) - r (C 2) b (p a +1 ,t) =ÿ(1)(n - 1ÿ+ t - a) - (n - 2)(0)ÿ= n - 1ÿ+ t - a. Therefore we conclude that q 1 = n - 2ÿ+ t - a . There are two paths p a,t from v a to v t . They are an (n - 2ÿ+ t - a, 1) -path and an (n - 1ÿ+ t - a, 1) -path.

Considering the (n - 2ÿ+ t - a, 1) -path and (5) we have q 2 = r (C 1) b (p a,t) - b (C 1) r (p a,t) =ÿ(n - 1)(1) - (1)(n - 2ÿ+ t - a)ÿ= a - t +ÿ1 . Considering the (n - 1ÿ+ t - a, 1) -path and (5) we have q 2 = r (C 1) b (p a,t) - b (C 1) r (p a,t) =ÿ(n - 1)(1) - (1)(n - 1ÿ+ t - a)ÿ= a - t. Therefore, we conclude that q 2 = a - t . Now by Lemma 2.2

we have [h l] = M [q 1 q 2] = M [n - 2ÿ+ t - a a - t] = [n 2 - 3 n +ÿ2ÿ+ t - a n - 2] , and hence k v a ,v a +1 (v t) = n 2 - 3 n + t - a (6) Universal Journal of Applied Mathematics 2(6): 250-255, 2014 253 for all 1 = t = a . Case 2 . The vertex w = v t where a +ÿ1 = t = n There is a unique path p a +1 ,t from v a +1 to v t which is a (t - a - 1 , 0) -path.

Using this path and (4) we have q 1 = b (C 2) r (p a +1 ,t) - r (C 2) b (p a +1 ,t) =ÿ(1)(

t - a - 1) - (n - 2)(0)ÿ= t - a - 1 . There is a unique path p a,t from v a to v t which is a (t - a - 1 , 1) -path. Using this path and (5) we have q 2 = r (C 1) b (p a,t) - b (C 1) r (p a,t) =ÿ(n - 1)(1) - (1)(t - a - 1)ÿ= n - t + a. By Lemma 2.2

we ?nd that [h l] = M [q 1 q 2] = M [t - a - 1 n - t + a] = [n 2 - 3 n +ÿ1ÿ+ t - a n - 1] , and hence k v a ,v a +1 (v t) = n 2 - 3 n + t - a (7) for all a +ÿ1 = t = n . From (6) and (7) we conclude that k v a ,v a +1 (W (2) n) = n 2 - 2 n +ÿ1 - a and by (2) we have k (W (2) n) = n 2 - 2 n +ÿ1 - a . It remains to show that k (W (2) n) = n 2 - 2 n +ÿ1 - a . For each vertex v t ,ÿt =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , we show that there is v t (h,l) -? v 1 with [h l] = [n 2 - 3 n +ÿ3 - a n - 2] . By Proposition 2.1

it suf?ces to show that the system M z + [r (p t, 1) b (p t, 1)] = [n 2 - 3 n +ÿ3 - a n - 2] (8) has nonnegative integer solution for some path p t, 1 from v t to v 1 . The solution to the system (8) is the integer vector z = [(n - 1 - a)ÿ+ÿ(n - 2) b (p t, 1) - r (p t, 1) a - 1ÿ+ r (p t, 1)ÿ+ b (p t, 1) - b (p t, 1) n] . If 1 = t = a , then there is an (n - t, 1) -path p t, 1 from v t to v 1 . Using this path we have that z 1 = n - 3ÿ+ t - a and z 2 = a - t .

Since t = 1 and a = n - 2 we have z 1 = 0 and since t = a we have z 2 = 0 . If a +ÿ1 = t = n , then there is an (n - t, 0) -path p t, 1 from v t to v 1 . Using this path we have that z 1 = t - (a +ÿ1) and z 2 = n - t + a - 1 . Since t = a +ÿ1 we have z 1 = 0 and since t = n and a = 1 we have z 2 = 0 . Thus for each t =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , there is a path p t 1 from v t to v 1 such that the system (8) has nonnegative integer solution. By Proposition 2.1

for each vertex v t ,ÿt =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , there is an (h,ÿl) -walk from v t to v 1 with h = n 2 - 3 n +ÿ3 - a and l = n - 2 . We now can conclude that for each pair of distinct vertices v i and v j in W (2) n , there is vertex v 1 with the property that there are v i (h,l) -? v 1 walk and v j (h,l) -? v 1 walk with [h l] = [n 2 - 3 n +ÿ3 - a n - 2] . This implies k (W (2) n) = n 2 - 2 n +ÿ1 - a .

We next discuss the scrambling index of primitive two- colored Wielandt digraph that contains two blue arcs. We ?rst discuss the case where the two blue arcs have the same terminal vertex. Theorem 3.3 Let W (2) n be a two colored Wielandt digraph on n = 4 vertices. If W (2) n has two blue arcs v n - 1 ? v 1 and v n ? v 1 , then k (W (2) n)ÿ= n 2 - 2 n +ÿ1 . Proof. We ?rst show that k (W (2) n) = n 2 - 2 n +ÿ1 .

It suf?ces to show that k v n ,v 1 (W (2) n) = n 2 - 2 n +ÿ1 . We assume there are v n (h,l) -? w and v 1 (h,l) -? w walks for some vertex w in W (2) n . We set up a lower bound for k v n ,v 1 (w) . Notice that for each t =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , there is a unique path p 1 ,t from v 1 to v t which is a (t - 1 , 0) -path and there is a unique path p n,t from v n to v t which is a (t - 1 , 1) -path.

Using the (t - 1 , 0) -path from v 1 to v t and (4) we have q 1 = b (C 2) r (p 1 ,t) - r (C 2) b (p 1 ,t) =ÿ(1)(t - 1) - (n - 2)(0)ÿ= t - 1 . Using the (t - 1 , 1) -path from v n to v t and (5) we have q 2 = r (C 1) b (p n,t) - b (C 1) r (p n,t) =ÿ(n - 1)(1) - (1)(t - 1)ÿ= n - t. Now Lemma 2.2 implies that [h l] = M [q 1 q 2] = M [t - 1 n - t] = [n 2 - 3 n +ÿ1ÿ+ t n - 1] . Therefore k v n ,v 1 (v t) = n 2 - 2 n + t for all 1 = t = n .

Since t = 1 , we conclude that k v n ,v 1 (W (2) n) = n 2 - 2 n +ÿ1 and by (2) we have k (W (2) n) = n 2 - 2 n +ÿ1 . We next show that k (W (2) n) = n 2 - 2 n +ÿ1 . We show that for each t =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , there is a v t (h,l) -? v 1 walk with [h l] = [n 2 - 3 n +ÿ2 n - 1] . By Proposition 2.1

it suf?ces to show that the system of equa- tion M z + [r (p t, 1) b (p t, 1)] = [n 2 - 3 n +ÿ2 n - 1] (9) has a nonnegative integer solution for some path p t, 1 from v t to v 1 . The solution to the system (9) is the integer vector z = [(n - 2) b (p t, 1) - r (p t, 1) n - 1ÿ+ r (p t, 1)ÿ+ b (p t, 1) - b (p t, 1) n] .

If 1 = t = n - 1 , then there is a (n - 1 - t, 1) -path p t, 1 from v t to v 1 . Using this path we ?nd that z 1 = t - 1 and z 2 = n - 1 - t . Since t = 1 we have z 1 = 0 and since t = n - 1 we have z 2 = 0 . If t = n , there is a (0 , 1) -path p n, 1 from v n to v 1 . Using this path we have z 1 = n - 2 and z 2 =ÿ0 . Thus for each t =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , there is a path p t 1 from v t to v 1 such that the system (9) has nonnegative integer solution.

254 The Scrambling Index of Two-colored Wielandt Digraph By Proposition 2.1 for each vertex v t ,ÿt =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , there is an (h,ÿl) -walk from v t to v 1 with h = n 2 - 3 n +ÿ2 and l = n - 1 . Therefore, for each pair of distinct vertices v i and v j there is vertex v 1 with the property that there exist v i (h,l) -? v 1 walk and v j (h,l) -? v 1 walk with [h l] = [n 2 - 3 n +ÿ2 n - 1] . Therefore, we conclude that k (W (2) n) = n 2 - 2 n +ÿ1 .

The following theorem presents the scrambling index of primitive two-colored Wieland digraph with two blue arcs that have the same initial vertex. Theorem 3.4 Let W (2) n be a two colored Wielandt digraph on n = 4 vertices. If W (2) n has two blue arcs v n - 1 ? v 1 and v n - 1 ? v n , then k (W (2) n)ÿ= n 2 - 2 n +ÿ2 . Proof. We show that k (W (2) n) = n 2 - 2 n +ÿ2 .

It suf?ces to show that k v n - 1 ,v n (W (2) n) = n 2 - 2 n +ÿ2 . For this purpose we assume that there are v n (h,l) -? w and v n - 1 (h,l) -? w for some w ? W (2) n . We set up a lower bound fro k v n - 1 ,v n (w) and consider two cases depending on the position of the vertex w . Case 1 .

The vertex w = v t where 1 = t = n - 1 There is a unique path p n,t from v n to v t which is a (t, 0) - path. Using this path and (4) we have q 1 = b (C 2) r (p n,t) - r (C 2) b (p n,t) =ÿ(1)(t) - (n - 2)(0)ÿ= t. There are two p n - 1 ,t paths from v n - 1 to v t . They are a (t - 1 , 1) -path and a (t, 1) -path.

Considering the (t - 1 , 1) -path and (5) we have q 2 = r (C 1) b (p n - 1 ,t) - b (C 1) r (p n - 1 ,t) =ÿ(n - 1)(1) - (1)(t - 1)ÿ= n - t. Considering the (t, 1) -path and (5) we have q 2 = r (C 1) b (p n - 1 ,t) - b (C 1) r (p n - 1 ,t) =ÿ(n - 1)(1) - (1)(t)ÿ= n - t - 1 . Hence we conclude that q 2 = n - t - 1 . Now Lemma 2.2 implies that [h l] = M [q 1 q 2] = M [t n - t - 1] = [n 2 - 3 n +ÿ2ÿ+ t n - 1] .

Thus k v n - 1 ,v n (v t) = n 2 - 2 n +ÿ1ÿ+ t (10) for all 1 = t = n - 1 . Case 2 . The vertex w = v n There is a (n - 1 , 1) -path from v n to v n . Using this path and (4) we ?nd that q 1 = b (C 2) r (p n,n) - r (C 2) b (p n,n) =ÿ(1)(n - 1) - (n - 2)(1)ÿ=ÿ1 . There is a (0 , 1) -path from v n - 1 to v n .

Using this path and (5) we ?nd that q 2 = r (C 1) b (p n - 1 ,n) - b (C 1) r (p n - 1 ,n) =ÿ(n - 1)(1) - (1)(0)ÿ= n - 1 . Now Lemma 2.2 implies that [h l] = M [q 1 q 2] = M [1 n - 1] = [n 2 - 2 n +ÿ1 n] . Thus k v n - 1 ,v n (v n) = n 2 - n +ÿ1 . (11) By considering (10) and (11) we conclude that k v n - 1 ,v n (W (2) n) = n 2 - 2 n +ÿ2 and by (2) we conclude k (W (2) n) = n 2 - 2 n +ÿ2 .

We next show that k (W (2) n) = n 2 - 2 n +ÿ2 . For each vertex v t ,ÿt =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , we show that there is v t (h,l) -? v 1 walk with [h l] = [n 2 - 3 n +ÿ3 n - 1] . By Proposition 2.1 it suf?ces to show that the system of equa- tions M z + [r (p t, 1) b (p t, 1)] = [n 2 - 3 n +ÿ3 n - 1] (12) has a nonnegative integer solution for some path p t, 1 from v t to v 1 .

The solution to the system (12) is the integer vector z = [1ÿ+ÿ(n - 2) b (p t, 1) - r (p t, 1) n - 2ÿ+ r (p t, 1)ÿ+ b (p t, 1) - b (p t, 1) n] . If 1 = t = n - 1 , then there is a (n - t, 1) -path p t, 1 from v t to v 1 . Using this path we ?nd z 1 = t - 1 and z 2 = n - 1 - t . Since t = 1 we have z 1 = 0 , and since t = n - 1 we have z 2 = 0 . If t = n , then there is a (1 , 0) -path p n, 1 from v n to v 1 .

Using this path we have z 1 =ÿ0 and z 2 = n - 1 . Therefore, for each t =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , the system (12) has a nonnegative integer solution for some path p t, 1 from v t to v 1 . By Proposition 2.1 for each vertex v t ,ÿt =ÿ1 , 2 ,ÿ.ÿ.ÿ.ÿ,ÿn , there is an (h,ÿl) -walk from v t to v 1 with h = n 2 - 3 n +ÿ3 and l = n - 1 .

We now conclude for each pair of distinct vertices v i and v j there is vertex v 1 with the property that there are v i (h,l) -? v 1 and v j (h,l) -? v 1 walks with [h l] = [n 2 - 3 n +ÿ3 n - 1] . Hence k (W (2) n) = n 2 - 2 n +ÿ2 . Let S W (2) n n denote the set of positive integers k for which there exists a primitive two-colored Wielandt digraph with scrambling index equals to k .

The following result gives the characterization for the set S W (2) n n . Corollary 3.5 Let W (2) n be a primitive two-colored Wielandt digraph on n = 4 vertices. Then S W (2) n n = { k : n 2 - 3 n +ÿ3 = k = n 2 - 2 n +ÿ2 } . Universal Journal of Applied Mathematics 2(6): 250-255, 2014 255 Proof. We note from Theorem 3.2 that [n 2 - 3 n +ÿ3 ,ÿn 2 - 2 n] ? S W (2) n n since 1 = a = n - 2 . By Theorem 3.3 and Theorem 3.4

we conclude that [n 2 - 3 n +ÿ3 ,ÿn 2 - 2 n +ÿ2] ? S W (2) n n . Since there are only n distinct primitive two-colored Wielandt digraphs on n vertices, we have S W (2) n n =ÿ[n 2 - 3 n +ÿ3 ,ÿn 2 - 2 n +ÿ2] . REFERENCES [1] M. Akelbek, S. Kirkland. Coef?cients of ergodicity and the scrambling index, Linear Algebra and its Applications, 430, 1111–1130, 2009. [2] M. Akelbek, S. Kirkland.

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